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Thermodynamics in an Expanding Universe:

The universe at early times consist of elementary particles. It can be considered as a thermal bath where the relevant degrees of freedom have equilibrium distribution function.

An important difference arises between an expanding universe and a thermodynamic system that has a fixed size. In the latter case, the system reaches thermal equilibrium if one waits sufficiently long (provided that interactions exist). In an expanding universe, however, thermal equilibrium at all times is not guaranteed. Even if all particles are in equilibrium at a given time, they will not necessarily be so later. The main difference arise because of the expansion. The rate for interactions must be sufficiently fast such that an equilibrium distribution can track the Hubble expansion. In other words, the interaction rate must be faster than the expansion rate. We will discuss this in more detail later on.

Assuming thermal equilibrium, the distribution function of a speies is given by:

Boson $\rightarrow$ $$f^{BE}_{(\vec{p})} = \frac{1}{\exp\left(\frac{E}{T}\right) - 1}$$ (Bose-Einstein distribution)

$$E = \sqrt{|\vec{p}|^2 + m^2}$$

Fermion $\rightarrow$ $$f^{FD}_{(\vec{p})} = \frac{1}{\exp\left(\frac{E}{T}\right) - 1}$$ (Fermi-Dirac distribution)

Here T is the temperature of thermal bath, E is the energy, $|\vec{p}|$ is the physical momentum, and m is the mass of that particle. We assume zero chemical potential and work in the natural units where k (Boltzmann constant) $= 1$.

Note that for any particle species, the other particles in the universe play the role of heat reservoir.

In the limit that $E \gg T$, the two distributions (which take into account of quantum effects for identical particles) are reduced to the familiar Maxwell-Boltzmann distribution:

$$f^{MB}_{(\vec{p})} = \exp\left(\frac{-E}{T}\right)$$

The (average) number density n, energy density ρ, and pressure p from a given particle species follows;

$$n = \frac{g}{(2\pi)^3} \int f_{(\vec{p})}\, d^3p$$

$$\rho = \frac{g}{(2\pi)^3} \int E\, f_{(\vec{p})}\, d^3p$$

$$p = \frac{g}{(2\pi)^3} \int \frac{|\vec{p}|^2}{3E}\, d^3p$$

Here "g" is the number of internal degrees of freedom associated with a species. For example, g=2 in the case of photon (the two polarizatio states of electromagnetic waves). Also, in the case of electrons g=4: two spin states (up and down) for the electron, and two spin states fo its antiparticle "positron". In relativistic quantum theory (its consistent formulation is "quantum field theory") particles and antiparti are accompanied by each other. Antiparticles have the same quantum numbers as particles except for charge.

Now lets consider two limits: (1) Relativis limit, where $T \gg m$, (2) Non relativistic limit where $T \ll m$.

(1) In the relativistic limit the thermal bath essentially consists of massless particles. Then:

$$\rho = \begin{cases} \frac{\pi^2}{30} g_B T^4 & \text{Bosons} \\ \left(\frac{7}{8}\right) \frac{\pi^2}{30} g_F T^4 & \text{Fermions} \end{cases}$$

$$n = \begin{cases} \frac{\zeta(3)}{\pi^2} g_B T^3 & \text{Bosons} \\ \left(\frac{3}{4}\right) \frac{\zeta(3)}{\pi^2} g_F T^3 & \text{Fermions} \end{cases}$$

$$p = \frac{1}{3} \rho$$

Here $\zeta(3) = 1.20206\ldots$ is the Riemann Zeta function of 3. $\left(\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \right)$.

Note that $\rho \propto T^4 \, (\propto a^{-4})$ and $n \propto T^3 \, (\propto a^{-3})$, where "a" is the scale factor of the universe.

(2) In the non relativistic limit the thermal bath consists of particles with small kinetic energy, and we have:

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(-\frac{m}{T}\right)$$ Bosons & Fermions

$\rho = m n$

$p = nT \ll \rho$

As expected, $p \ll \rho$. Also, because $E \simeq m \gg T$, we have the same expression for bosons and fermions.

The expressions for ρ, n, p when $T \sim m$ are more complicated.

An important point to note is that having any particles in the non relativistic regime at the present time can happen only if that particle is not in thermal equilibrium.

For example, lets consider the electron. Its mass is $m_e \simeq 0.5$ MeV, while temperature at the present time is $T_0 \sim 3 \times 10^{-4}$ eV. The exponential factor $\exp\left(-\frac{m_e}{T_0}\right) \sim \exp(-10^9)$ results in an extremely huge suppression in the number density of the electrons today. However, from neutrality of the universe we know that the number of electrons must be at least equal to the number of protons, whose primordial value is ~ 0.4 protons per m^3.

Actually, as we will see, particles drop out of thermal equilibrium after temperature of the universe drops below their mass. The reason being that the only way number density can track the exponentially suppressed equilibrium value for $T \ll m$ is for particles and antiparticles efficiently annihilate each other (this cannot happen for particles or antiparticles only due charge conservation). However, as the universe expands the mean distance between particles and antiparticles increases, and hence they cannot find each other as easily. This implies that at some point annihilation rate becomes smaller than the expansion rate of the universe (it "freezes"). Henceforth, the number density of non relativistic particles and antiparticles is subject to Hubble expansion and scales $\propto a^{-3}$ (or T^3, where T is the temperature of the universe).

This essentially happens for all massive species. No matter how large T initially is, expansion of the universe lowers it, and at some point we have transition from the relativistic regime to the non-relativistic regime for a particle of mass m.

This explains why the number density of electrons is not incredibly small today. However, there are two more questions in this regard: Why the electron number density is not smaller (as can be estimated from the freeze out of annihilation)? Why aren't there positrons in significant number today?

(There are positrons from cosmic rays, but their number is far smaller than the number of electrons seen in the universe.)

These questions are actually related to each other and have to do with the observed fact that the universe is made of matter as far as we can see. This is the so-called "matter-antimatter asymmetry" in the universe. We will discuss this in more depth, and possible ways to generate such an asymmetry, later in this course.

In general, we can define the total number of massless degrees of freedom in the thermal bath:

$$\rho = \frac{\pi^2}{30} g_* T^4$$

$$g_* = \sum_B g_B + \frac{7}{8} \sum_F g_F$$

The sum is over all bosonic (B) and fermionic (F) degrees of freedom. For example, in a plasma of photons, electrons and positrons (as happens at temperatures > 1 MeV) we find $g_* = 2 + \frac{7}{8} \times 4 = \frac{11}{2}$.

A useful quantity is the entropy density s. From the first law of thermodynamics:

$$dU = p\,dV - T\,dS$$

where $U = \rho V$ and $S = sV$. At constant V we have

$dU = -T\,dS$, which results in:

$$ds = -\frac{1}{T}\,d\rho \Rightarrow s = \frac{2\pi^2}{45} g_* T^3 \qquad \left(g_* = \sum_B g_B + \frac{7}{8}\sum_F g_F\right)$$

The importance of s is that the total entropy S is constant as long as the processes are reversible (adiabatic expansion). This turns out to be the case in most parts of the evolution of the universe. The notable epoch where the total entropy increases dramatically is during "reheating" after inflation where all particles are created. This happens due to out-of-equilibrium decay of a Bose condensate (the inflaton field) that drove inflation. There could be other out of equilibrium decays in the early universe. Other than these instances the evolution is largely adiabatic throughout the history of the early universe.

If the total entropy $S = sV$ is constant, then the entropy density s scales as a^{-3}. As a result, normalizing the number density of any particle species n_i by s removes the effect of the expansion (note that expansion changes any number density $\propto a^{-3}$). Thus $\frac{n_i}{s}$ will not change because of the Hubble expansion. It can be considered as a "comoving number density" that is constant unless microphysical processes (decay or annihilation as examples) change it.